Math 102

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September 18, 2018

Announcements

- Diagnostic Test
  - See my email for more information
  - Online course switching closes today
- OSH Groups
  - Only one person submits the OSH for the whole group. (If you were in a group and your OSH graded didn't appear, let me know.)
  - If you want to work as a group for OSH 2, you will need to make the group again. And then again for OSH 3, etc...
  - If you had technical issues please let me know so that I can report them.
- Office hours - will remain as-is

Last Time

- Let $f(x)$ be a function, and let x_0 be a number. The instantaneous rate of change of $f(x)$ at $x = x_0$ is defined as the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

- The derivative of $f(x)$, denoted by $f'(x)$, is a new function defined by the property that, for any x_0,

$$f'(x_0) = \text{instantaneous rate of change of } f(x) \text{ at } x = x_0$$

Goals Today

- Limits
  - Gain intuition about what $\lim_{x \to a} g(x)$ means
  - How is $\lim_{x \to a} g(x)$ different from $g(a)$?
  - In which cases are these are equal? (Continuity)
  - If not, how do we compute the limit?
- The derivative $f'(x)$
  - When does it exist?
  - The graph of $f'(x)$ vs. the graph of $f(x)$

Limits

Let $g(x)$ be a function, and let a be a number.

$$\lim_{x \to a} g(x) = \text{‘\textit{The limit of} } g(x) \text{ \textit{as} } x \text{ \textit{goes to} } a.\text{’}$$

Limits

Let $g(x)$ be a function, and let a be a number.

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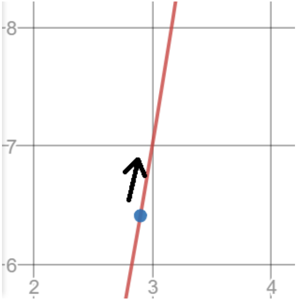
Intuitively - imagine that you are drawing the graph of $g(x)$. As your pencil moves towards $x = a$, what y-coordinate does it move towards?

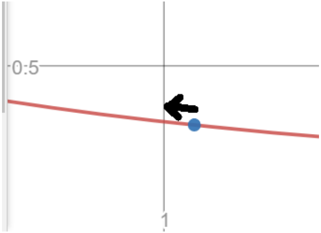
This is best understood through some graphical examples.

$\lim_{x \to a} g(x)$ - examples

- $\lim_{x \to 3} (x^2 - 2) = 7$



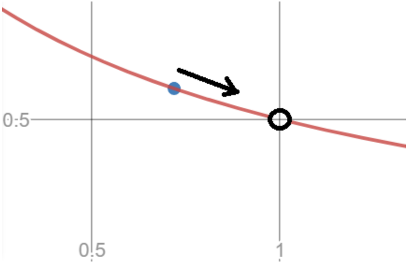
- $\lim_{x \to 1} \frac{1}{x+2} = \frac{1}{3}$



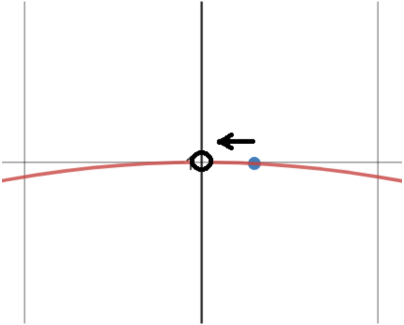
In both of these cases, there are ‘no surprises’ as x tends towards a.

$\lim_{x \to a} g(x)$ - examples

- $\lim_{x \to 1} \frac{x-1}{x^2-1} = \frac{1}{2}$



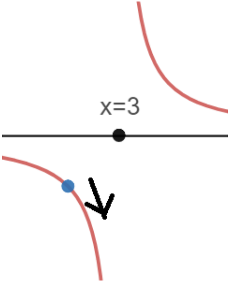
- $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$



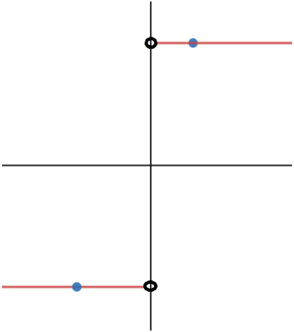
In these two cases, the function $g(x)$ is undefined at $x = a$. But $\lim_{x \to a} g(x)$ still exists.

$\lim_{x \to a} g(x)$ - examples

- $\lim_{x \to 3} \frac{1}{x-3}$ does not exist.



- $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

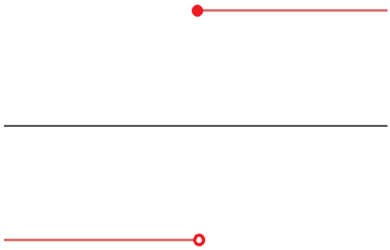


In these two cases, the function $g(x)$ is undefined at $x = a$ AND the limit does not exist.

$\lim_{x \to a} g(x)$ - examples

- It is possible that $g(a)$ exists but $\lim_{x \to a} g(x)$ does not.

$$g(x) = \begin{cases} 1 & x \geq 2 \\ -1 & x < 2 \end{cases}$$



- Question: Is it possible for $g(a)$ and $\lim_{x \to a} g(x)$ to both exist, but be unequal?

$\lim_{x \to a} g(x)$ - examples

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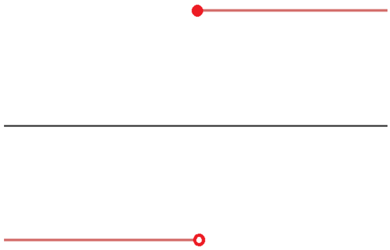

- Question: Is it possible for $g(a)$ and $\lim_{x \to a} g(x)$ to both exist, but be unequal? (yes, example on board)

Continuity

We say that $g(x)$ is continuous at a if the following three properties hold.

- $\lim_{x \to a} g(x)$ exists.
- $g(a)$ is defined.
- $\lim_{x \to a} g(x) = g(a)$.

Intuition - when you're drawing the graph of $y = g(x)$, you don't need to pick up your pencil at $x = a$.

$\lim_{x \to a} g(x)$ - examples

Exercise: Calculate $\lim_{x \to -3} \frac{x^2+4x+3}{x+3}$. (Hint: divide out a common factor)

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$$\begin{aligned}
\lim_{x \to -3} \frac{x^2 + 4x + 3}{x + 3} &= \lim_{x \to -3} \frac{(x+1)(x+3)}{x+3} \\
&= \lim_{x \to -3} (x+1) \\
&= -2
\end{aligned}$$

$\lim\limits_{x \to a} g(x)$ - examples

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Question: What does the graph of $y = \frac{x^2+4x+3}{x+3}$ look like?

The derivative

- Suppose that $f(x)$ is a function which is continuous at a. Zoom in on the graph of the function at $x = a$...

  https://www.desmos.com/calculator/wu6bcbuw2l

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- Suppose that $f(x)$ is a function which is continuous at a. Zoom in on the graph of the function at $x = a$...

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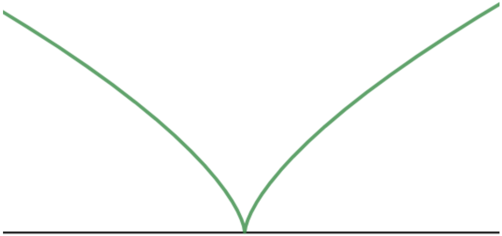
- If the graph starts to look like a line as we zoom in, then the derivative $f'(x)$ is well-defined at $x = a$. We say $f(x)$ is differentiable at $x = a$.

Continuous, but not differentiable at $x = a$

If a function is differentiable, then it has to be continuous. But it's possible to be continuous and NOT differentiable. The phenomenom below is known as a cusp.



Question: True or False? If $\lim_{x \to a} f(x)$ exists, then $f(x)$ is continuous at $x = a$.

Question: True or False? If $f(x)$ is continuous at $x = a$, then $f'(a)$ exists.

Question: True or False? If $\lim_{x \to a} f(x)$ exists, then $f(x)$ is continuous at $x = a$. False.

Question: True or False? If $f(x)$ is continuous at $x = a$, then $f'(a)$ exists. False.

Question: Which of the following implies that $f'(x)$ is defined at $x = x_0$?

(A) $f'(x)$ is differentiable at $x = x_0$.

(B) $\frac{f(x+h)-f(x)}{h}$ is continuous at $x = x_0$.

(C) $\frac{f(x_0+h)-f(x_0)}{h}$ is continuous at $h = 0$.

(D) $\frac{f(x_0+h)-f(x_0)}{h}$ is continuous at $x = x_0$.

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(C) $\frac{f(x_0+h)-f(x_0)}{h}$ is continuous at $h = 0$.

(D) $\frac{f(x_0+h)-f(x_0)}{h}$ is continuous at $x = x_0$.

(Explanation on board)

$f'(x_0)$ is defined as the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Think of x_0 as a fixed number and h as a variable. Consider the function $g(h) = \frac{f(x_0+h)-f(x_0)}{h}$. Then we are precisely asking whether $\lim_{h \to 0} g(h)$ exists. (C) says that $g(h)$ is continuous at $h = 0$, which is an even stronger statement and certainly implies that $\lim_{h \to 0} g(h)$ exists.

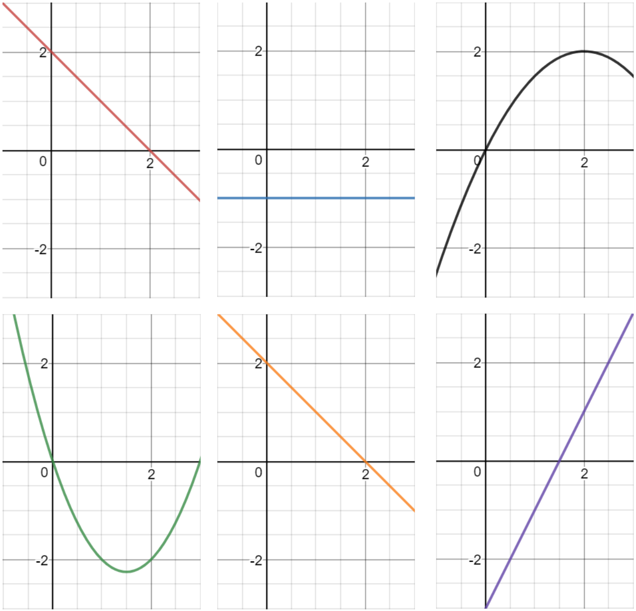
(credit to the person who pointed this out) (A) also is correct for the reason that if the derivative $f'(x)$ is differentiable at $x = x_0$, then it has to be continuous at $x = x_0$. Then it has to have a value at $x = x_0$.

The graph of $y = f(x)$ vs. $y = f'(x)$.

- Example (on board): $f(x) = 3x$.

- Exercise: If $f(x) = |x|$, sketch the graph of $f'(x)$.

Match $y = f(x)$ to $y = f'(x)$!



Answers

$$\begin{array}{c|c|c} f'(x) & g'(x) & f(x) \\ h(x) & g(x) & h'(x) \end{array}$$

Note that $g(x) = f'(x)$ in this case!

Recap

- $g(a)$ vs. $\lim_{x \to a} g(x)$
- When the function is continuous at $x = a$, these are equal
- Calculating some limits
- Continuous vs. Differentiable
- Graph Sketching (intro)